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Gwenaël Piaser

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IPAG Business School
184, Boulevard Saint-Germain
75006 Paris
France

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Common Agency Games with Common Value Exclusion, Convexity and Existence

Gwenaël Piaser†

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Abstract

We consider the model common agency proposed by Biais Martimort and Rochet (2000, 2013). We show that in this setting there is no symmetric equilibrium as the one characterized in those articles. We argue that the equilibrium price schedules cannot be simultaneously convex and concave. In particular in the monopoly case, under some classical assumptions, some agents will be excluded from trade. In the other that a price schedule at any symmetric equilibrium must be must be convex and concave. We conclude that a symmetric equilibrium cannot exist and discuss the implications of our result and the links with the existing literature.

*I would like to thank Tristan Boyer for his help. All errors are mines and copyrighted.
†IPAG Business School, Paris
1 Introduction

Biais, Martimort and Rochet (2000, 2013), BMR thereafter, consider a multi-principals game to analyze imperfect competition under adverse selection in financial markets. Strategic liquidity suppliers post nonlinear prices (such as limit order schedules) which stand ready to trade with a risk-adverse agent who has private information on the fundamental value of the asset as well as on his hedging needs. BMR show that there exists an unique equilibrium in convex schedules and they analyze its properties.

In the following we argue that such an equilibrium does not exist. We provide a two steps argument. First, in the third section, we consider the monopolist case. In this context we show equilibrium price schedule is never convex (nor concave). We show that the optimal price schedule cannot be simultaneously continuous, convex (or concave) and satisfies the “no distortion at the top” property. The main difference with the traditional model à la Maskin and Riley (1984) is that quantities can be positive or negative. In other word he monopolist can buy or sell the asset. In this framework, we cannot neglect that some agents are excluded: They do not trade with the monopolist. We show that the optimal price schedule is not necessarily continuous and can be convex (or concave) only on some parts.

In a second step, treated in the forth section of the present paper, we consider the multiple principal case. We consider a non-exclusive competition: Agents can deal with more than one principal, even if it is not compulsory. In this framework we show two things. First, we provide a characterization of the BMR’s equilibrium. We discuss the properties of such equilibrium and show that principals offer continuous and convex price schedules. In the other hand, we show that those schedules at equilibrium must satisfied a “no distortion at the top” property, stated in the proper way in a competition setting. Using a argument similar to the one given in the monopolistic framework. We argue that convexity and the “no distortion at the top” property are incompatible. Hence, we conclude that the characterized equilibrium does not exist.

The main proofs are geometrical, based on basic properties of convex functions and thus are quite simple. Our main concern is about “exclusion” which is often neglected. In common agency games it can be a critical issue concerning the existence of some symmetric equilibria. In the conclusion, we discuss our negative result in the line of the existing literature on common agency. Contrarily to the BMR’s approach most of
the existing paper consider a finite number of types for the agents. In the latter models, existence is problematic and when an equilibrium exists, it is not symmetric in the way we define symmetry in the present paper. Hence one can see our results as first stone of a bridge between BMR’s model and discrete type models.

2 Model

We consider a common agency game, denoted $G^n$, of incomplete information. There are $(n+1)$ players in the game, $n$ principals and one agent. The principals play first, they offer simultaneously “mechanisms”. A “mechanism” is a mapping from a message space ($M_i$ is the set of all possible message spaces for principal $i$, $i \in \{1, \ldots, n\}$) to the decision space. Here a principal takes two decisions, a price $T_i$ and a quantity $q_i$, the decision space is $\mathbb{R}^2$. The principal $i$’s preferences over $q_i$ and $T_i$ are represented by the following profit function:

$$
\pi(q_i, T_i, \theta) = T_i - \nu(\theta) q_i.
$$

The function $\nu$ is continuous and differentiable, weakly increasing $\nu \geq 0$.

The agent’s preferences are represented by a quadratic utility function.

$$
U\left((q_i, T_i)_{i=1, \ldots, n}, \theta\right) = \theta \sum_{i=1, \ldots, n} q_i - \frac{\gamma \sigma^2}{2} \left( \sum_{i=1, \ldots, n} q_i \right)^2 - \sum_{i=1, \ldots, n} T_i.
$$

The variables $\gamma$ and $\sigma$ are common knowledge. The variable $\theta$ is known only by the agent, principals know only the distribution of that variable over the range of possible values $\Theta \in [\underline{\theta}, \overline{\theta}]$. The density function is denoted $f$, and the repartion function of the same variable is denoted $F$. The functions $f$ and $F$ exhibit the usual regularity properties. These functions are common knowledge.

In this setting optimal quantities, those that would be chosen by an informed benevolent planner, are given by:

$$
\forall \theta \in [\underline{\theta}, \overline{\theta}] \quad q^*(\theta) = \frac{\theta - \nu(\theta)}{\sigma^2 \gamma}.
$$

We will assume that $q^*(\underline{\theta}) < 0$ and $q^*(\overline{\theta}) > 0$. Principals, if they were perfectly informed, would like to buy share from some agents and to sell share we other agents.
This feature makes the model different from those considered in general in Industrial Organization.

The game follows a classical timing. First each principal $i$ offers a (measurable) mapping from $M_i$ to $\mathbb{R}^2$ denoted $(T_i(.), q_i(.))$. We denoted by $\Gamma_i$ set of such mappings. The agent can either reject or accept the offer. If he accepts then he sends the message $m_i \in M_i$ (we must have $m_i \in M_i$), the agent gets from principal $i$ the decision $(T_i(m_i), q_i(m_i))$. In the BMR model the interpretation of $(T_i(m_i), q_i(m_i))$ is the following: The agent must trade the quantity $q_i(m_i)$ at the price $T_i(m_i)$. If the agent rejects the offer from principal $i$, he gets $(0,0)$ from him. The agent observes all the offered mechanisms and he decides to reject or accept some of them. We formalize participation by considering that the agent faces the sets of messages $\mathcal{M}_i = M_i \cup \{\emptyset\}$, and we imposes that for any principal $i$ and any mechanism in $\Gamma_i$, $T_i(\emptyset) = 0$, and $q_i(\emptyset) = 0$. Then a strategy, denoted $\lambda$, for the agent if a mapping from the set $\times_{i=1,...,n} \Gamma_i \times \Theta$ to the set $\times_{i=1,...,n} \mathcal{M}_i$.

Following standard analyses of competing mechanism games, we focus on Perfect Bayesian Equilibrium as the relevant solution concept.

3 Monopolist

Since we consider the case with only one principal, we give up the subscript $i$. The monopolist want to set the functions $q$ and $T$ in order to maximises its profit. First, by the Revelation Principle we can solve the monopolist problem as if $M = \Theta$, and imposing incentive compatibility for the mapping $(T(\cdot), q(\cdot))$. From a direct mechanism, we can derive a price schedule (denoted $t$) which is a more realistic mechanism though indirect. Such a price schedule is defined by

$$\forall \theta \in [\theta, \overline{\theta}], \quad t(q(\theta)) = T(\theta).$$

First we characterize the optimal incentive compatible mechanism. Then, we will discuss the properties of optimal price schedule $t$. Since we impose incentive compatibility, the optimal direct mechanism must satisfy the following conditions:
The first order condition

$$\theta \dot{q}(\theta) - \gamma \sigma^2 q(\theta) \dot{q}(\theta) - T(\theta) = 0$$

(5)

or equivalently

$$\dot{U}(\theta) = q(\theta),$$

(6)

where the function $U$ is defined by

$$U(\theta) \equiv \theta q(\theta) - \frac{\gamma \sigma^2}{2} [q(\theta)]^2 - T(\theta).$$

(7)

Second order condition:

$$\dot{q} \geq 0,$$

(8)

or equivalently

$$\ddot{U} \geq 0.$$

(9)

Thus the monopolist maximizes

$$\int_{\theta \in [\theta, \tilde{\theta}]} [T(\theta) - v(\theta) q(\theta)] dF(\theta)$$

(10)

with respect to $T$ and $q$ under the constraints $U \geq 0, \dot{U} = q$ and $\dot{q} \geq 0$.

The conditions $\dot{U}(\theta) = q(\theta)$ and $\ddot{U} \geq 0$ imply that the function $U$ is decreasing when $q$ is negative, increasing when $q$ is positive, and constant when $q$ is equal to zero. Moreover, since $U$ is convex, $U$ is constant and equal to zero on a connected subset of $[\theta, \tilde{\theta}]$. We denote by $[\theta^m, \theta^m_u]$ the subset of $[\theta, \tilde{\theta}]$ on which $U(\theta) = 0$, or equivalently on which $q(\theta) = 0$. Let us assume that $q_m(\theta) < 0$ and $q_m(\tilde{\theta}) > 0$, then the solution has the following form:

Over the set $[\theta, \theta^m]$ the monopolist maximizes

$$\int_{\theta \in [\theta, \theta^m]} [T(\theta) - v(\theta) q(\theta)] dF(\theta)$$

(11)

with respect to $T$ and $q$ under the same constraints: $U \geq 0, \dot{U} = q$ and $\dot{q} \geq 0$. 5
To solve the problem first we observe that

\[ T(q) = \theta q(\theta) - \frac{\gamma \sigma^2}{2} [q(\theta)]^2 - \mathcal{U}(\theta), \quad (12) \]

and since \( \mathcal{U}(\theta^n) = 0 \), we can state that

\[ \mathcal{U}(\theta) = \int_{\theta^n}^{\theta} q(\delta) d\delta \quad (13) \]

and integrating by parts

\[ \int_{\theta \in \Theta^n} \mathcal{U}(\theta) dF(\theta) = -\int_{\theta \in \Theta^n} q(\theta) \frac{F(\theta)}{f(\theta)} dF(\theta). \quad (14) \]

We neglect the constraint \( \dot{q} \geq 0 \), and hence we consider that the monopolist maximizes

\[ \int_{\theta \in \Theta^n} \left[ \theta q(\theta) - \frac{\gamma \sigma^2}{2} [q(\theta)]^2 + q(\theta) \frac{F(\theta)}{f(\theta)} - \nu(\theta) q(\theta) \right] dF(\theta) \quad (15) \]

with respect to the function \( q \). A pointwise optimization gives \( q_m(\theta) = q^*(\theta) + \frac{F(\theta)}{\gamma \sigma^2 f(\theta)} \), where \( q^*(\theta) = \frac{\theta - \nu(\theta)}{\gamma \sigma^2} \). Finally, let us remark that the function

\[ \theta q(\theta) - \frac{\gamma \sigma^2}{2} [q(\theta)]^2 + q(\theta) \frac{F(\theta)}{f(\theta)} - \nu(\theta) q(\theta) \quad (16) \]

is concave with respect to \( q(\theta) \), hence we have characterized a maximum.

Over the set \( (\theta^n, \Theta) \), the optimal quantity is \( q_m(\theta) = q^*(\theta) - \frac{1-F(\theta)}{\gamma \sigma^2 f(\theta)} \).

The optimal quantity \( q \) with respect to the type as the following form:

- If \( \theta \in [\theta, \theta^n] \) then \( q_m(\theta) = q^*(\theta) + \frac{F(\theta)}{\gamma \sigma^2 f(\theta)} \),
- if \( \theta \in (\theta^m_n, \Theta) \) then \( q_m(\theta) = 0 \),
- if \( \theta \in (\theta^m_n, \Theta) \) then \( q_m(\theta) = q^*(\theta) - \frac{1-F(\theta)}{\gamma \sigma^2 f(\theta)} \).

Hence, if \( \theta^m_n \) and \( \Theta^n \) are given, the quantities \( T(\theta^m_n) \) and \( T(\theta^m_n) \) are given by the equations

\[ T(\theta^m_n) = \theta^m_n q(\theta^m_n) - \frac{\gamma \sigma^2}{2} [q(\theta^m_n)]^2, \quad (17) \]
and
\[ T(\theta^m_a) = \theta^m_a q(\theta^m_a) - \frac{\gamma \sigma^2}{2} [q(\theta^m_a)]^2. \]

Hence, for a given couple \((\theta^m_a, \theta^m_b)\), the function \(T\) is fully characterized.

From that, it follows that one can defined the profit function \(\Pi(\theta^m_a, \theta^m_b)\) where \((\theta^m_a, \theta^m_b) \in [\hat{\theta}, \bar{\theta}]^2\) and one can characterize the optimal couple \((\theta^m_a, \theta^m_b)\) by maximizing \(\Pi(\theta^m_a, \theta^m_b)\), with respect to \((\theta^m_a, \theta^m_b)\). The optimal choice of \(\theta^m_b\) do not influence the optimal choice of \(\theta^m_a\) (and vice versa), then the optimizations with respect to \(\theta^m_b\) and \(\theta^m_a\) can be done separately.

We want \(q(\theta^m_b) \geq 0\) and \(q(\theta^m_b) \leq 0\). We define \(\hat{\theta}\) as the type such that
\[ q^*(\hat{\theta}) - \frac{1 - F(\hat{\theta})}{\gamma \sigma^2 f(\hat{\theta})} = 0, \tag{19} \]

i.e. the type \(\hat{\theta}\) is such that \(q(\hat{\theta}) = 0\), and is the minimal value for \(\theta^m_b\). In the same way, we can define \(\hat{\theta}\) as the type such that \(q^*(\hat{\theta}) + \frac{F(\hat{\theta})}{\gamma \sigma^2 f(\hat{\theta})} = 0\) and interpret \(\hat{\theta}\) as the maximal value for \(\theta^m_b\). Hence the function \(\Pi(\theta^m_a, \theta^m_b)\) is defined and continuous over \([\theta, \hat{\theta}] \times [\hat{\theta}, \bar{\theta}]\), and hence \(\theta^m_b\) is well defined. Since \(\hat{\theta} < \bar{\theta}\), we have \(\theta^m_a < \theta^m_b\).

Now, we want to analyze the price schedule \(t\)’s properties. From what precedes, we can find conditions under which the price schedule \(t\) is concave or convex at any point \(q\). From the agent’s first order condition we have:
\[ \hat{T}(\theta) = \theta \hat{q}(\theta) - \gamma \sigma^2 q(\theta) \hat{q}(\theta), \tag{20} \]

which can be rewritten
\[ t'[q(\theta)] \hat{q}(\theta) = \theta \hat{q}(\theta) - \gamma \sigma^2 q(\theta) \hat{q}(\theta) \tag{21} \]

and hence
\[ t''[q(\theta)] \hat{q}(\theta) = \hat{\nu}(\theta) - \frac{d}{d\theta} \left( \frac{1 - F(\theta)}{\gamma \sigma^2 f(\theta)} \right). \tag{22} \]
If \( \theta \in (\theta_a^m, \theta_b^m) \) and \( \tau'' \{ q(\theta) \} \dot{q}(\theta) = \ddot{\nu}(\theta) + \frac{d}{d\theta} \left( \frac{F(\theta)}{\gamma \sigma^2 f(\theta)} \right) \) and if \( \theta \in [\theta_a^m, \theta_b^m] \) then implies that the schedule at the point \( q(\theta) \), where \( \theta \in (\theta_a^m, \theta_b^m) \) is strictly convex if (and only if)

\[
\ddot{\nu}(\theta) > \frac{d}{d\theta} \left( \frac{1 - F(\theta)}{1 - \frac{F(\theta)}{\gamma \sigma^2 f(\theta)}} \right),
\]

and similarly, if for any \( \theta \in [\theta_a^m, \theta_b^m] \) we have \( \dot{\nu}(\theta) > -\frac{d}{d\theta} \left( \frac{F(\theta)}{\gamma \sigma^2 f(\theta)} \right) \) then the price schedule \( t \) is locally convex around \( q(\theta) \). It is then natural to define the following condition:

**Definition 1**  Condition 1 is satisfied if

- \( \forall \theta \in [\theta_a^m, \theta_b^m], \dot{\nu}(\theta) > -\frac{d}{d\theta} \left( \frac{F(\theta)}{\gamma \sigma^2 f(\theta)} \right) \).
- \( \forall \theta \in (\theta_a^m, \theta_b^m), \ddot{\nu}(\theta) > \frac{d}{d\theta} \left( \frac{1 - F(\theta)}{1 - \frac{F(\theta)}{\gamma \sigma^2 f(\theta)}} \right) \).

At a given point \((q(\theta), T(\theta))\), the schedule \( t \) is convex if the derivative \( \dot{\nu}(\theta) \) is sufficiently high. Intuitively if the difference between the valuation of the monopolist and the agent gets bigger when \( \theta \) increases, then the monopolist tries to take more and more surpluses from the agent as his type gets higher.

Condition 1 must be carefully interpreted, in particular, if the price schedule can be convex on some subset of \([\theta_a^m, \theta_b^m]\), it cannot be globally convex.

**Theorem 1**  If condition 1 is satisfied then the price schedule \( t \) is not continuous.

**Proof.** We consider the space \((q, T)\). Since we assumed that \( q^+(\theta) < 0 \) and \( q^-(\theta) > 0 \), Then \( t[q^+(\theta)] < 0 \) and \( t[q^-(\theta)] > 0 \).

We must have that \( t[q^+(\theta)] - \nu(\theta) q^+(\theta) > 0 \) and \( t[q^-(\theta)] - \nu(\theta) q^-(\theta) > 0 \).

Due to the definition of \( q^+(\theta) \), it turns out that the price schedule \( t \) is tangent to the line \( \Pi(\theta) \) defined by the equation

\[
T - \nu(\theta) q = t[q^+(\theta)] - \nu(\theta) q^+(\theta)
\]

at the point \((q^+(\theta), t[q^+(\theta)])\). In the same way, the price schedule \( T \) is tangent to the line defined by the equation:

\[
T - \nu(\theta) q = t[q^-(\theta)] - \nu(\theta) q^-(\theta)
\]
at the point \((q^*(\theta), t[q^*(\theta)]\)).

If we assume that the schedule \(t\) is convex and continuous, then \(t\) is above the line \(\Pi(\theta)\). Moreover, \(q^*(\theta) > 0\). Then it exists a \(\theta \in [\theta, \bar{\theta}]\) such that \(q^m(\theta) = 0\) and \(t[q^m(\theta)] > 0\), which is impossible. 

The theorem shows that there is a incompatibility between the “no distortion at the top” property, convexity and continuity of \(t\). The “no distortion at the top” property and the convexity of \(t\) implies that \(t\) is above the profit line \(\Pi(\bar{\theta})\). This line is itself above the point \((0,0)\) or cross that point. Since \(t\) is strictly above this profit line, it is also strictly above \((0,0)\), meaning that at equilibrium there is an agent paying price strictly positive price for nothing, which is clearly impossible.

A similar argument applies is we consider concavity rather than convexity.

**Definition 2** Condition 2 is satisfied if

- \(\forall \theta \in [\theta, \theta_d]\), \(\dot{\nu}(\theta) < -\frac{d}{d\theta} \left(\frac{F(\theta)}{\nu^2 f(\theta)}\right)\).
- \(\forall \theta \in (\theta_d, \bar{\theta}]\), \(\dot{\nu}(\theta) < \frac{d}{d\theta} \left(\frac{1-F(\theta)}{\nu^2 f(\theta)}\right)\).
In this case (discount prices) one can exclude that the optimal solution is continuous.

**Theorem 2** If condition 2 is satisfied then the price schedule $t$ is not continuous.

**Proof.** At the point $(q^*(\theta), t[q^*(\theta)])$, the price schedule $t$ is tangent to the line $\Pi(\theta)$, or in other words $\frac{\partial t}{\partial q} |_{(q^*(\theta), t[q^*(\theta)])} = v(\theta)$.

If the schedule $t$ is continuous and concave then

$$\frac{\partial t}{\partial q} |_{(q^*(\theta), t[q^*(\theta)])} > \frac{\partial t}{\partial q} |_{(q^*(\theta), t[q^*(\theta)])}.$$  

From the characterization of $t$ we have $\frac{\partial t}{\partial q} |_{(q^*(\theta), t[q^*(\theta)])} = v(\theta)$. Since the function $v$ is increasing, $v(\theta) \geq v(\theta)$. A contradiction.  

An optimal mechanism, if either condition 1 or condition 2 is satisfied, divides the set of possible agents into three categories. Agents with a “hight” $\theta$. As those agents have a high valuation of the asset, principal sells assets to them. For agents with a “low” $\theta$, as their valuation is low, the principal buys assets from them. There a third category:
agents with a “medium” valuation. From them any trade is not profitable. This set is not a singleton because discrimination is costly for the principal. Hence there is a non degenerate of set of agents for which exchanges with the principal are not interesting.

From the two previous theorems, we can state the following corollary:

**Corollary 1** If the price schedule \( t \) is continuous, then it is neither convex, nor concave.

It is worth to note that if the price schedule might be discontinuous, the equilibrium utility is continuous with respect to the agent’s type. In formal terms \( U \) is a continuous function. Such a property is a direct consequence of the continuity of the utility function \( U \) with respect to \( \theta \).

Concerning the monopolist case, our remarks helps to precised the shape of the optimal price schedule. It turns out that the potential discontinuity that we have stressed in the monopoly case, can be problematic for the existence of equilibria in non-exclusive competition games.

## 4 Non-Exclusive Competition

In our setting of non-exclusive contracting the Revelation Principle does not apply anymore (Peck, 1997). We follow the existing literature on common agency and we apply the “Delegation Principle” (see Peters, 2001; Martimort and Stole, 2002). Nevertheless we impose more restrictions on the shape of the equilibrium. In particular we want to restrict attention to symmetric equilibria. In particular at equilibrium we consider that every principal offers the same price schedule, and the agent buy the same quantity from all principals, and hence pays to them the same total price. More specifically, let us denote by \( t_i \) the equilibrium price schedule offered by principal \( i \).

**Definition 3** An equilibrium is fully symmetric if all principals offer the same menus of contracts: \( \forall i, j \in \{1, 2, \ldots, n\}, \ t_i = t_j = t, \) and when the agent as the type \( \theta \), he buy the quantity \( q(\theta) \) from all the principals.

Hence, we consider equilibria in which the agent (when is type is \( \theta \)) owns the quantity \( nq(\theta) \) and pays \( nt(q(\theta)) \).
Let us remark that is an equilibrium is fully symmetric, then principal are offering menus (or price schedule here) without “latent contract”. By “latent contracts” we mean contract (or couples price quantity) that are never bought at equilibrium. The role of such contracts is widely studied in the literature in different contexts (see Hellwig, 1983; Bisin and Guaitoli, 2004; Ales and Maziero, 2009; Piaser, 2010; Attar et al., 2011, 2013)

We define the curve (or the set) $\zeta$ as the set of all consumption bundle

$$
\zeta = (n q(\theta); nt(q(\theta)))_{\theta \in [\theta_A, \theta_B]}.
$$

(27)

**Definition 4** We said that the equilibrium is monotone is the curve $\zeta$ is increasing.

It is natural to consider that the total price paid is increasing with the quantity bought. Since we concentrate on the simplest equilibria, restricting attention to monotone equilibria do not appear problematic.

**Lemma 1** If the equilibrium is fully symmetric and monotone, the curve $\zeta$ is convex in the space $(q, T)$

**Proof.** Let us consider two point of $\zeta$, namely $A = (n q(\theta_A); nt(\theta_A))$ and $B = (n q(\theta_B); nt(\theta_B))$, and we assume that between $A$ and $B$ the curve $\zeta$ is continous and above the segment $[A, B]$. Let us consider the point $C$ defined as $C = ((n - 1) q(\theta_A) + q(\theta_B); (n - 1) t(\theta_A) + t(\theta_B))$. By construction the point $C$ belongs to the segment $[A, B]$. Since indifference curves are concave, it implies that it exists a type $\tilde{q} \in [\theta_A, \theta_B]$ such that the agent having this type, strictly prefer the point (or contract) $C$ to any point of the curve $\zeta$.

If now, keeping the same notation, we consider the point $A$ and $B$ of the curve $\zeta$ such that for any quantity $q^n \in [n q(\theta_A), n q(\theta_B)]$ there is no $T^n$ such that $(q^n, T^n) \in \zeta$. Roughly speaking, we assume that the curve $\zeta$ is discontinuous between the point $A$ and $B$. Since the curve $\zeta$ is the set of optimal choices, and since preferences exhibit the single crossing property, it must exist a type $\tilde{\theta}$ such that the agent having the type $\tilde{\theta}$ is indifferent between $A$ and $B$. Using the same argument, one can construct a point $C$ lying on the segment $[A, B]$. By construction the contract $C$ is feasible and strictly preferred to the contracts $A$ and $B$ by the agent having the type $\tilde{\theta}$.
\[ T = q(B) + q(A) \]

Figure 3: Convexity of the curve \( \zeta \)

\[ (q(\theta_B); T(\theta_B)) \]

\[ C = (q(\theta_B) + q(\theta_A); T(\theta_B) + T(\theta_A)) \]

\[ (q(\theta_A); T(\theta_A)) \]

Figure 4: Continuity of the curve \( \zeta \)

\[ A = (q^2(\theta_A); T^2(\theta_A)) \]

\[ C = (q(\theta_A); T(\theta_A)) \]

\[ B = (0, 0) \]
To conclude, the curve $\zeta$ is continuous and convex.

The convexity of $\zeta$ do not rely on assumption on the function $v$ and $F$ as in the monopoly case. Here convexity comes from a simple arbitrage argument. Convexity is a direct consequence of competition.

These first results do not provide a characterization of the equilibrium of the game $G^n$. In the next theorem, we argue that the equilibrium price schedules are monotone and exhibits the “no distortion at the top” properties. Since a proof of this theorem is given by BMR, we just sketch a proof in appendix.

Theorem 3 The equilibrium quantities $n q(\Theta)$ and $n q(\overline{\Theta})$ are not distorted and equal to $q^*(\Theta)$ and $q^*(\overline{\Theta})$.

Hence, the equilibrium schedule $\zeta$ must be convex (and continuous). Using the same arguments as in the monopolist case, we can argue or the geometrical impossibility of a continuous and convex schedule. From that, we can state our last theorem

Theorem 4 It does not exist any fully symmetric equilibrium in the game $G^n$.

5 Conclusion

First let us remark that our non-existence problem comes from the restriction to pure strategy equilibria. If we consider mixed strategy equilibria, then Carmona and Fajardo (2009) and Monteiro and Page Jr (2008) show the existence of a Nash equilibrium in common agency games such as the one considered in the present paper. Characterization of such equilibria remains an open problem.

In the existing literature, common agency games with common values do not have fully symmetric equilibria, for example Attar et al. (2011), Ales and Maziero (2009) or Attar et al. (2013). Those papers study model with a finite number of types. Attar et al. (2011) show that in a model with linear preferences equilibria latent contracts play a crucial role. Ales and Maziero (2009) and Attar et al. (2013) in model with risk averse agents extend the latter paper’s results. Moreover, in their framework, at equilibrium some agents depending on their type, are excluded from trade. Hence our paper a first step toward a clarification of the links between the BMR approach and the
existing results in games with a discrete number of types. Moreover, in a model of complete information, e.i. a degenerate version of the considered model, Chiesa and Denicolò (2009) show that at equilibrium at least one principal is offering a menu with latent contract. This suggest that in common agency games with incomplete information latent contracts are likely to play an important role.

Let us also insist on the importance of exclusion. In the literature on mechanism design, it has been largely neglected, the paper by Jullien (2000) being one of the few exceptions. In the our context it cannot be seen as a sophistication of the equilibrium as it is the case in the mechanisms design literature. Since we have at equilibrium principals trade positive and negative quantities, exclusion plays a fundamental role. A principal, in most situation, would like to divide the agents into three groups. Those to which he sells, those from which he buys, and a “middle” group: agent from any trade is not profitable. Due to asymmetric information, discrimination is costly. Thus, excluding some agents from the trade, those with valuations close to the one of the principal, helps him to sells at higher prices and buy a lower prices. Such a strategy implies that optimal price schedule is discontinuous. In the paper we show that competition implies continuous price schedule when we restrict attention to some natural equilibria. Since a price schedule cannot be simultaneously continuous and discontinuous, we have a problem of existence.

Except theorem 3, all our results do not depend on the shape of the agent’s preferences, as long as those preferences are sufficiently regular (continuity, concavity, etc). Theorem 3 just generalized a basic property of equilibrium schedules, which is standard in the mechanism design literature. Hence we conjecture that our results remains valid if we consider agents with more general quasi-linear preferences.

Finally, the shape of equilibria in common agency games with both common values and an continuum of type remains an open question. Papers studying similar models with a discrete number of types such as Ales and Maziero (2009) and Attar et al. (2013) characterize equilibria in which only one type of agent trades. Moreover, some restricting assumptions are needed. In our context, if such an equilibrium exists, it would mean an almost complete collapse of the market. Existence and characterization of an equilibrium in the present context could be a relevant track for future researches.
A Proof of Theorem 3

We consider the principal \textit{i} and we consider that all other principals plays the same convex price schedule \textit{t}_j. Following Martimort and Stole (2002), we consider that there is no loss of generality in restricting principal \textit{i} to offer incentive compatible mechanisms.

Hence, first we consider that principal \textit{i} offers direct mechanisms i.e; a mappings \((q_i(\cdot), T_i(\cdot))\) from \([\theta, \bar{\theta}]\) to \(\mathbb{R}^2\). If the agent \(\theta\) reports the vector \(\hat{(\theta_i, (q_j)_{j\neq i})}\), that is if he reports \(\hat{\theta}_i\) to principal \(i\), and ask for the quantities \(q_j\) to principals \(j\), he gets:

\[
U\left(\hat{\theta}_i, (q_j)_{j\neq i}\right|\theta) = \theta \left(q_i(\hat{\theta}_i) + \sum_{j\neq i} q_j\right) - \frac{\gamma \sigma^2}{2} \left(q_i(\hat{\theta}_i) + \sum_{j\neq i} q_j\right)^2 - T_i(\hat{\theta}_i) - \sum_{j\neq i} t_j(q_j).
\]

We focus on principal \(i\). He considers others principals’ strategies \((t_j)_{j\neq i}\) as given. By \(Q_j \subset \mathbb{R}^2\) we denote the support of the price schedule \(t_j\).

We denote by \((q_j)_{j\neq i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)\), the set of reports sent by the agent to the other principals. We define the best reports \((\hat{q}_j(\theta, \hat{\theta}_i))_{j\neq i}\) given the type (which is \(\theta\)) of the agent and given an arbitrary report to principal \(i\) denoted \(\hat{\theta}_i\).

\[
(\hat{q}_j(\theta, \hat{\theta}_i))_{j\neq i} \in \arg\max_{(q_j)_{j\neq i} \in \times_{j\neq i} Q_j} \left\{ U\left(\hat{\theta}_i, (q_j)_{j\neq i}\right|\theta) \right\}.
\]

We look for an interior the solution, then those quantities must satisfy:

\[
\frac{\partial U\left(\hat{\theta}_i, (q_j(\theta, \hat{\theta}_i))_{j\neq i}\right|\theta)}{\partial q_1, \ldots, \partial q_{i-1}, \partial q_{i+1}, \ldots, \partial q_n} = 0,
\]

which can be also written

\[
\forall j \neq i \quad \theta - \gamma \sigma^2 \left(q_i(\theta) + \sum_{j\neq i} q_j(\theta, \hat{\theta}_i)\right) = t_j'(q_j(\theta, \hat{\theta}_i)).
\]

Since the price schedule \(t_j\) are convex and the agent preferences are concave with respect to the quantity, then second order conditions are straightforwardly satisfied.
Now, we derive the optimal strategy of the principal $i$. As we apply the Revelation Principle, we assume the principal $i$ offers incentive compatible mechanisms. In other words, principal $i$ offers maps $T_i(\cdot), q_i(\cdot)$ such that the agent reports to him his true type $\theta$, given the price schedules offered by the other principals.

The agent reports truthfully his type to principal $i$ if

$$\frac{dU}{d\tilde{\theta}_i} \bigg|_{\tilde{\theta}_i = \theta} = 0,$$

applying the envelope theorem we get an equivalent expression:

$$\theta q_i(\theta) - \gamma \sigma^2 \left( \sum_{j \neq i} q_j(\theta) + q_i(\theta) \right) q_i(\theta) - T_i(\theta) = 0. \quad (33)$$

To simplify the expressions, we used the notation $q_j(\theta, \tilde{\theta}_j) = q_j(\theta)$. We now can define the rent obtained by the agent. The rent is the utility that the agent gets if his type is $\theta$ given the offers made by all principals

$$U(\theta) = \theta \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) - \frac{\gamma \sigma^2}{2} \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right)^2 - T_i(\theta) - \sum_j t_j(q_j(\theta)), \quad (34)$$

Applying again the envelope theorem, we get the derivative of $U$ with respect to $\theta$:

$$\forall \theta \in [\underline{\theta}, \overline{\theta}], \quad \frac{dU}{d\theta} = q_i(\theta) + \sum_{j \neq i} q_j(\theta). \quad (35)$$

To get that, a necessary condition is $\frac{\partial^2 U(\theta)}{\partial \theta \partial \theta} < 0$. If messages are optimal when $\tilde{\theta}_i = \theta$ then the conditions $\forall \theta \in [\underline{\theta}, \overline{\theta}], \quad \frac{\partial^2 U(\theta)}{\partial \theta \partial \theta} < 0$ becomes

$$\forall \theta \in [\underline{\theta}, \overline{\theta}], \quad \theta \tilde{q}_i(\theta) - \gamma \sigma^2 \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) \tilde{q}_i(\theta) - \gamma \sigma^2 \tilde{q}_i^2(\theta) - T_i(\theta) < 0. \quad (36)$$

Using standard methods of mechanism design, this last condition can also be written as
In words, the optimal quantity must be non decreasing with $\theta$. This condition is standard in mechanism design theory.

If the function $q_i(.)$ is increasing, it obviously means that it can be first negative, then equal to zero and finally positive. We denote $[\theta, \theta_a]$ the domain on which the function $q_i(.)$ is negative, $[\theta_a, \theta_b]$ on which $q_i(.)$ is constant and equal to zero and $[\theta_b, \bar{\theta}]$ on which it is positive. Using these new notation, we can integrate the function $\mathcal{U}$ to get a new expression of $\mathcal{U}$.

\[
\mathcal{U} (\theta) = \int_{\theta}^{\bar{\theta}} \left( q_i (\theta) + \sum_{j \neq i} q_j (\theta) \right) d\theta \quad (38)
\]

if $\theta \in [\theta_b, \bar{\theta}]$,

\[
\mathcal{U} (\theta) = \int_{\theta}^{\theta_a} \left( q_i (\theta) + \sum_{j \neq i} q_j (\theta) \right) d\theta \quad (39)
\]

if $\theta \in [\theta, \theta_a]$, and

\[
\mathcal{U} (\theta) = 0. \quad (40)
\]

if $\theta \in [\theta_a, \theta_b]$, where $\theta \leq \theta_a \leq \theta_b \leq \bar{\theta}$.

Integrating by parts these expressions gives

\[
\int_{\theta}^{\theta_a} \mathcal{U} (\theta) dF (\theta) = \int_{\theta}^{\theta_a} \left( q_i (\theta) + \sum_{j \neq i} q_j (\theta) \right) \frac{F (\theta)}{f (\theta)} dF (\theta) \quad (41)
\]

and

\[
\int_{\theta_b}^{\bar{\theta}} \mathcal{U} (\theta) dF (\theta) = \int_{\theta_b}^{\bar{\theta}} \left( q_i (\theta) + \sum_{j \neq i} q_j (\theta) \right) \frac{1 - F (\theta)}{f (\theta)} dF (\theta). \quad (42)
\]

The profit of principal $i$ can be written as

\[
\Pi = \int_{[\theta, \bar{\theta}]} [T_i (\theta) - v (\theta) q_i (\theta)] dF (\theta), \quad (43)
\]

by using the definition of the utility function we can rewrite the former expression as:
\begin{align*}
\Pi &= \int_{\Theta_{b}} \left[ \theta \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) - \frac{\sigma^2}{2} \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right)^2 \right] df(\theta) \\
&\quad - \int_{\Theta_{a}} \left[ \mathcal{U}(\theta) + \sum_{j \neq i} t_j(q_j(\theta)) + v(\theta) q_i(\theta) \right] df(\theta) \\
&\quad + \int_{\Theta_{b}} \left[ \theta \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) - \frac{\sigma^2}{2} \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right)^2 \right] df(\theta) \\
&\quad - \int_{\Theta_{b}} \left[ \mathcal{U}(\theta) + \sum_{j \neq i} t_j(q_j(\theta)) + v(\theta) q_i(\theta) \right] df(\theta).
\end{align*}

(44)

The first order condition of the Principal’s maximization problem is given by

\begin{align*}
\theta \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) - \frac{\sigma^2}{2} \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right)^2 \\
- \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) \left( \frac{1-F(\theta)}{f(\theta)} \right) - \sum_{j \neq i} t_j(q_j(\theta)) - v(\theta) q_i(\theta).
\end{align*}

(45)

The second order condition is given by

\begin{align*}
\theta \left( 1 + \sum_{j \neq i} \frac{\partial q_j(\theta)}{\partial q_i(\theta)} \right) - \gamma \sigma^2 \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) \left( 1 + \sum_{j \neq i} \frac{\partial q_j(\theta)}{\partial q_i(\theta)} \right) \\
- \left( 1 + (n-1) \sum_{j \neq i} \frac{\partial q_j(\theta)}{\partial q_i(\theta)} \right) \frac{1-F(\theta)}{f(\theta)} - \sum_{j \neq i} \left[ t_j'(q_j(\theta)) \frac{\partial q_j(\theta)}{\partial q_i(\theta)} \right] - v(\theta) = 0.
\end{align*}

(46)
\[ \sum_{j \neq i} \frac{\partial^2 q_{ji}(\theta)}{\partial q_i(\theta)^2} \gamma^2 \left( 1 + \sum_{j \neq i} \frac{\partial q_{ji}(\theta)}{\partial q_i(\theta)} \right)^2 - \gamma^2 \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) \sum_{j \neq i} \frac{\partial^2 q_{ji}(\theta)}{\partial q_i(\theta)^2} \]

\[-(n-1) (1 - \gamma F(\theta)) \sum_{j \neq i} \frac{\partial^2 q_{ji}(\theta)}{\partial q_i(\theta)^2} - \sum_{j \neq i} \left[ t_{j'}''(q_j(\theta)) \left( \frac{\partial q_j(\theta)}{\partial q_i(\theta)} \right)^2 \right] \]

\[-\sum_{j \neq i} \left[ t_{j'}'(q_j(\theta)) \frac{\partial^2 q_j(\theta)}{\partial q_i(\theta)^2} \right] \leq 0. \]

As we want to show that a fully symmetric equilibrium does not exist, we will assume that this condition is satisfied.

To characterize the solution we need the expression of \( \sum_{j \neq i} \frac{\partial q_{ji}(\theta)}{\partial q_i(\theta)} \). From the self-selection constraint, we have derived the expressions:

\[ q_{gs}^2 \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) = t_{j'}'(q_j(\theta)) \frac{\partial^2 q_j(\theta)}{\partial q_i(\theta)^2}, \]

(48)

Over the set \( [\theta_b, \theta] \), the function \( q_i(\cdot) \) is increasing. Thus, without loss of generality, we can rewrite the direct mechanism \( (q_j(\cdot), T_j(\cdot)) \) as a direct mechanism \( (q_j(\cdot), t_j(q_j(\cdot))) \). As we consider a domain in which \( q_j(\cdot) \) is strictly increasing, we can define a inverse function \( \theta_j^{-1}(\cdot) \). Thus the function \( t_j \) is define for all \( \theta \) in \( [\theta_b, \theta] \) by \( t_j(q(\theta)) = T_j \left( \theta_j^{-1}(q(\theta)) \right) \). The former first order condition (31) becomes:

\[ \forall j \neq i \quad \theta - \gamma^2 \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) = t_j'(q_j(\theta)). \]

(49)

Differentiating this equation with respect to \( q_i(\theta) \) gives (as \( q_i(\theta) \) is a parameter in (31), this transformation makes sense):

\[ -\gamma^2 \left( 1 + \sum_{j \neq i} \frac{\partial q_j(\theta)}{\partial q_i(\theta)} \right) = t_j''(q_j(\theta)) \frac{\partial q_j(\theta)}{\partial q_i(\theta)}. \]

(50)

By summing these conditions over \( j \neq i \), we get:
\[-(n-1)\gamma \sigma^2 \left(1 + \sum_{j \neq i} \frac{\partial q_j(\theta)}{\partial q_i(\theta)}\right) = \sum_{j \neq i} t''_j(q_j(\theta)) \frac{\partial q_j(\theta)}{\partial q_i(\theta)}. \quad (51)\]

As we consider a symmetric equilibrium all the principals \(j\), (with \(j\) different from \(i\)), are offering the same mechanism, and thus the derivative \(t''_j(q_j(\theta))\) and the quantity \(q_j(\theta)\) are constant with respect to \(j\) and we denote them \(t''(q(\theta))\) and \(q(\theta)\). Thus we get:

\[-\frac{(n-1)\gamma \sigma^2}{t''(q(\theta))} + (n-1)\gamma \sigma^2 = (n-1) \frac{\partial q(\theta)}{\partial q_i(\theta)}. \quad (52)\]

Equation (49) at equilibrium can be written

\[\theta - n\gamma \sigma^2 q(\theta) = t'(q(\theta)). \quad (53)\]

Differentiating this equation with respect to \(\theta\) gives

\[1 - n\gamma \sigma^2 \dot{q}(\theta) = t''(q(\theta)) \dot{q}(\theta), \quad (54)\]

or

\[\frac{1}{\dot{q}(\theta)} - n\gamma \sigma^2 = t''(q(\theta)). \quad (55)\]

Thus at equilibrium we have,

\[(n-1) \frac{\partial q(\theta)}{\partial q_i(\theta)} = -(n-1) \frac{\dot{q}(\theta) \gamma \sigma^2}{1 - \dot{q}(\theta) \gamma \sigma^2}. \quad (56)\]

From (49) we get

\[t'(q(\theta)) = \theta - n\gamma \sigma^2 q(\theta). \quad (57)\]

Using the two obtained expressions, from the first order condition (46) can get a equilibrium condition:
\[
\theta \left( 1 - \frac{\dot{q}(\theta)}{1 - \gamma \sigma^2 q(\theta)} \right) \gamma \sigma^2 \frac{(n - 1)}{\gamma \sigma^2 n q(\theta) \left( 1 - \frac{\dot{q}(\theta)}{1 - \gamma \sigma^2 q(\theta)} \right)} \gamma \sigma^2 (n - 1) \\
- \left( 1 - \frac{\dot{q}(\theta)}{1 - \gamma \sigma^2 q(\theta)} \right) \frac{(1 - F(\theta))}{F(\theta)} \\
+ (n - 1) \left[ \theta - \gamma \sigma^2 n q(\theta) \right] \frac{\dot{q}(\theta)}{1 - \gamma \sigma^2 q(\theta)} \gamma \sigma^2 - v(\theta) = 0.
\]

Consequently, using the notation \( q^* (\theta) = \frac{\theta - v(\theta)}{\gamma \sigma^2} \) and \( q_m(\theta) = q^* (\theta) - \frac{1 - F(\theta)}{\gamma \sigma^2 f(\theta)} \), we get:

\[
[q_m(\theta) - n q(\theta)] - \dot{q}(\theta) \gamma \sigma^2 [q_m(\theta) - n q(\theta)] \\
- [q_m(\theta) - n q(\theta)] \dot{q}(\theta) \gamma \sigma^2 (n - 1) \\
+ (n - 1) [q^* (\theta) - n q(\theta)] \dot{q}(\theta) \gamma \sigma^2 = 0,
\]

and finally

\[
\dot{q}(\theta) = \frac{1}{\gamma \sigma^2} \left( n + \frac{(n - 1) (q^* (\theta) - n q(\theta))}{n q(\theta) - q_m(\theta)} \right)^{-1},
\]

and if we define the function \( q^n(\cdot) = n q(\cdot) \), we get equivalently

\[
\dot{q}^n(\theta) = \frac{1}{\gamma \sigma^2} \left( 1 + \frac{(n - 1) (q^* (\theta) - q^n(\theta))}{n (q^n(\theta) - q_m(\theta))} \right)^{-1}.
\]

If \( \theta = \bar{\theta} \), since \( F(\bar{\theta}) = 1 \), equation (58) can be rewritten

\[
\bar{\theta} \left( 1 - \frac{\dot{\bar{q}}(\bar{\theta})}{1 - \gamma \sigma^2 q(\bar{\theta})} \gamma \sigma^2 (n - 1) \right) - \gamma \sigma^2 n q(\bar{\theta}) \left( 1 - \frac{\dot{\bar{q}}(\bar{\theta})}{1 - \gamma \sigma^2 q(\bar{\theta})} \gamma \sigma^2 (n - 1) \right) \\
+ (n - 1) \left[ \bar{\theta} - \gamma \sigma^2 n q(\bar{\theta}) \right] \frac{q(\bar{\theta})}{1 - \gamma \sigma^2 q(\bar{\theta})} \gamma \sigma^2 - v(\bar{\theta}) = 0,
\]

which is equivalent to

\[
\bar{\theta} - \gamma \sigma^2 n q(\bar{\theta}) - v(\bar{\theta}) = 0.
\]

Hence \( n q(\bar{\theta}) = q^*(\bar{\theta}) \).
If $\theta \in [\theta_L, \theta_U]$, the principal $i$ maximizes the following expression with respect to $q_i(\theta)$:

$$
\theta \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) - \frac{\sigma_i^2}{2} \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right)^2
$$

$$
- \left( q_i(\theta) + \sum_{j \neq i} q_j(\theta) \right) \frac{F(\theta)}{f(\theta)} + \sum_{j \neq i} t_j \left( q_j(\theta) \right) - v(\theta) q_i(\theta).
$$

(64)

We can derive the same expression for $\dot{q}(\theta)$, except that $q_m(\theta) = \frac{F(\theta)}{\theta^2 f(\theta)}$. Using the same argument as above, we can show that $nq(\theta) = q^*(\theta)$.

Given the expressions of $\dot{q}(\theta)$, $\theta_a$ and $\theta_b$ must be such that the function $q$ is continuous. As the aggregate supply $nq(.)$ is an increasing function, the form chosen for the utility is justified. Usual conditions on the density $f$ guaranty that $q$ is strictly increasing. Interpretation of these conditions are discussed by Miravete (2002).

Because the price schedule are continuous, at equilibrium we must have $nq(\theta_b) = 0$, with the function $q$ defined by equation (60). The same argument applies for $q(\theta_a)$.

Convexity of price schedules implies that any agent prefers a bundle belonging to $\zeta$ rather than a bundle made with the offers of $n - 1$ principals.

Finally, let us remark that one can get the same characterization if we assume that principals play incentive compatible mechanisms (see Piaser, 2007). This observation does not contradict the “Delegation Principle”. At equilibrium, principals do not offer contracts with “latent contracts”. Since we consider equilibria in which every principal offers a menu (which here is called a price schedule) without latent contract, then this equilibria can be characterized by incentive compatible mechanisms.

**References**


